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PLANE ELASTIC PROBLEM WITH INCOMPRESSIBILITY AND GEOMETRIC NONLINEARITY

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The plane problem is among the best-studied problems in the theory of elasticity. This is because the initial condition of the problem can be approximately realized in a number of cases of practical importance and because the problem can be studied using a complex analysis, which permits developing effective solution techniques to the investigation of the problem. Below, the plane problem is considered as applied to incompressible materials under geometrically nonlinear conditions.

Incompressible materials are widely used. Among them are, for example, a number of constructional as well as rubber-like and polymer materials. Incompressibility imposes a certain restriction on strain, and this leads to modification of the law of mechanical behavior.

Allowance for geometric nonlinearity in one or another form becomes necessary when the displacement gradients or their combinations can no longer be considered small over the entire body volume. This situation usually occurs in flexible bodies, in bodies with cavities near inner and outer boundaries, etc. Geometric nonlinearity modifies strain-displacement relations and gives rise to nonlinear terms in these relations.

Let us study the effect of incompressibility on nonlinear deformation within the framework of the plane problem using the theory of nonlinear elasticity developed by V. V. Novozhilov [1].

The static problem of elasticity includes relations between strains and elongation shears and rotations, representation of the latter quantities in terms of displacements, stress-strain relations, and equations of equilibrium subject to boundary conditions on the body surface. We represent these relations in terms of the actual state variables of the material.

In the actual variables, the stresses and strains are characterized by the symmetrical Cauchy P and Almansi ε tensors. The tensor ε is expressed in terms of the displacement vector $\mathbf{u}$ and the tensors e and ω, which are related to elongation shears and rotations by the following nonlinear formulas [2]:

$$2\varepsilon = \nabla\mathbf{u} + \mathbf{u}\nabla - (\nabla\mathbf{u})\cdot(\mathbf{u}\nabla) = 2e - e\cdot e + e\cdot\omega - \omega\cdot e + \omega\cdot\omega, \ \nabla\mathbf{u} = e + \omega, \ \mathbf{u}\nabla = e - \omega.$$

Here $\nabla\mathbf{u}$ and $\mathbf{u}\nabla$ are the displacement gradient and the transposed displacement gradient.

The theory of Novozhilov [1] assumes that the elongation-shear and rotation components $e_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are small in comparison with unity, and the former are of the same order as the product of the latter: $|e_{\alpha\beta}| \ll 1$ and $|\omega_{\alpha\beta}| \ll 1$, where $|e_{\alpha\beta}| \sim |\omega_{\sigma\nu}\omega_{\nu\tau}|$ (summation over twice-repeated subscripts is implied). By virtue of these assumptions, the strain tensor is given by an approximate nonlinear formula if only terms of the order of e are retained in the general expression of the tensor:

$$2\varepsilon = 2e + \omega\cdot\omega, \quad 2e = \nabla\mathbf{u} + \mathbf{u}\nabla, \quad 2\omega = \nabla\mathbf{u} - \mathbf{u}\nabla. \tag{1}$$

Consequently, in this theory, the strain components $\varepsilon_{\alpha\beta}$ will also be small quantities ($|\varepsilon_{\alpha\beta}| \ll 1$).

The Cauchy and Almansi tensors are related by Murnaghan's law [3]. This law follows from the energy balance equation applied to an isothermical virtual displacement $\delta\mathbf{x}$ of the medium in the actual state:

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$\rho dF/d\varepsilon : \delta\varepsilon = P : (\nabla\delta\mathbf{x})$, $F = F(\varepsilon)$ (ρ and F are the density of the material and the free energy). This equation written by means of the strain-tensor variation

$$2\delta\varepsilon = (\nabla\delta\mathbf{x})\cdot(G-2\varepsilon) + (G-2\varepsilon)\cdot(\delta\mathbf{x}\nabla) \tag{2}$$

(G is a metric tensor) in the form

$$\left[P - \rho(G-2\varepsilon)\cdot\frac{dF}{d\varepsilon}\right] : (\nabla\delta\mathbf{x}) = 0, \tag{3}$$

gives Murnaghan's law in the form

$$P = \rho(G-2\varepsilon)\,dF/d\varepsilon \tag{4}$$

for arbitrary independent components of the gradient $\nabla\delta\mathbf{x}$.

The density of the material is defined in terms of the initial density ρ_0 and the basic strain invariants ε_1, ε_2, and ε_3 by the formula [3]

$$\rho = \rho_0\sqrt{1-2\varepsilon_1+4\varepsilon_2-8\varepsilon_3} \quad [\varepsilon_1 = \operatorname{tr}\varepsilon, \quad 2\varepsilon_2 = (\operatorname{tr}\varepsilon)^2 - \operatorname{tr}\varepsilon^2, \quad \varepsilon_3 = \det\varepsilon]. \tag{5}$$

For an isotropic material, the free energy will also be a function of the strain invariants $F = F(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

For small strains, the invariants ε_1, ε_2, and ε_3 are small quantities of the first, second, and third order, respectively. In this case, the law (4) admits the approximate expression [4]

$$P = \rho_0\frac{dF}{d\varepsilon} + (\rho-\rho_0)\frac{dF}{d\varepsilon} - 2\rho\varepsilon\cdot\frac{dF}{d\varepsilon} \approx \frac{dF_*}{d\varepsilon}, \quad F_* = \rho_0 F. \tag{6}$$

For the quadratic representation of the free energy $F_* = (1/2)\lambda\varepsilon_1^2 + \mu(\varepsilon_1^2 - 2\varepsilon_2)$, λ and $\mu = \text{const}$ (λ and μ are the Lamé coefficients of elasticity) with allowance for the tensor gradients of strain invariants

$$\frac{d\varepsilon_1}{d\varepsilon} = G, \quad \frac{d\varepsilon_2}{d\varepsilon} = \varepsilon_1 G - \varepsilon, \quad \frac{d\varepsilon_3}{d\varepsilon} = \varepsilon_2 G - \varepsilon_1\varepsilon + \varepsilon^2, \tag{7}$$

relation (6) gives Hooke's law

$$P = \lambda\varepsilon_1 G + 2\mu\varepsilon. \tag{8}$$

The equation of equilibrium for a material with bulk-force density $\mathbf{f}$ has the form [2]

$$P\cdot\nabla + \mathbf{f} = 0. \tag{9}$$

Mixed conditions are usually specified on the body boundary: the displacement $\mathbf{h}$ on the part Σ_u of the surface and the stress $\mathbf{p}$ on the other part Σ_p:

$$\mathbf{u}\big|_{\Sigma_u} = \mathbf{h}, \quad P\cdot\mathbf{n}\big|_{\Sigma_p} = \mathbf{p}. \tag{10}$$

Here $\mathbf{n}$ is the unit vector of the outer normal to the body surface.

Equations (1), (8), and (9) and conditions (10) form the static problem for a compressible material in the geometrically nonlinear model of elasticity [5]:

$$\begin{gathered} P\cdot\nabla + \mathbf{f} = 0, \quad P = \lambda\varepsilon_1 G + 2\mu\varepsilon, \quad \varepsilon_1 = \operatorname{tr}\varepsilon, \quad \lambda = \text{const}, \quad \mu = \text{const}, \\ 2\varepsilon = 2e + \omega\cdot\omega, \quad 2e = \nabla\mathbf{u} + \mathbf{u}\nabla, \quad 2\omega = \nabla\mathbf{u} - \mathbf{u}\nabla, \quad \mathbf{u}\big|_{\Sigma_u} = \mathbf{h}, \quad P\cdot\mathbf{n}\big|_{\Sigma_p} = \mathbf{p}. \end{gathered} \tag{11}$$

For an incompressible material, these relations need modification.

Under incompressibility conditions, the initial dV_0 and actual dV volumes of a typical elementary particle are equal ($dV_0 = dV$). Therefore, the condition of conservation of mass under deformation ($\rho dV = \rho_0 dV_0$) leads to the coincidence of the initial and actual density ($\rho_0 = \rho$). In this case, relation (5) is simplified and is the incompressibility condition in the form of a constraint on the strain invariants:

$$\varepsilon_1 - 2\varepsilon_2 + 4\varepsilon_3 = 0. \tag{12}$$

In the case of small strains, Eq. (12) admits linearization:

$$\varepsilon_1 = 0. \tag{13}$$

If the material is incompressible, some components of the gradient $\nabla\delta\mathbf{x}$ are dependent and connected by one relation. This relation is established by varying (12) using expressions (2) and (7), and it has the form

$$\left(\frac{d\varepsilon_1}{d\varepsilon} - 2\,\frac{d\varepsilon_2}{d\varepsilon} + 4\,\frac{d\varepsilon_3}{d\varepsilon}\right) : \delta\varepsilon = [(1 - 2\varepsilon_1 + 4\varepsilon_2)G - 8(\varepsilon_2\varepsilon - \varepsilon_1\varepsilon^2 + \varepsilon^3)] : (\nabla\delta\mathbf{x}) = 0.$$

With allowance for (12) and the Hamilton–Kelley identity [6] for the strain tensor $\varepsilon^3 - \varepsilon_1\varepsilon^2 + \varepsilon_2\varepsilon - \varepsilon_3 G = 0$, this relation is simplified and takes the form

$$G : (\nabla\delta\mathbf{x}) = 0. \tag{14}$$

Thus, the constraint imposed by incompressibility on the virtual displacement gradient components does not depend on strain and is of the same form for both finite and small deformations.

To deduce the law of behavior of an incompressible material, we multiply condition (14) by the Lagrange multiplier q and add the result to relation (3) for this case. As a result, we obtain the equality $[P + qG - (G - 2\varepsilon)(dF_*/d\varepsilon)] : (\nabla\delta\mathbf{x}) = 0$, where F_* is defined by formula (6).

Let us use the arbitrariness of the multiplier q so that the coefficient of the dependent component of the gradient $\nabla\delta\mathbf{x}$ vanishes. Then, terms with independent components of the gradient will remain in the relation, and, by virtue of the arbitrariness of the components, their coefficients must become zero. As a result, the coefficients of all gradient components become zero, and this leads to the modified Murnaghan's law $P = -qG + (G - 2\varepsilon)(dF_*/d\varepsilon)$, where $F_* = F_*(\varepsilon)$. In this law, the Lagrange multiplier has the meaning of hydrostatic pressure; as to the free-energy arguments, they must be related by the incompressibility condition.

For small strains, the law assumes the form $P = -qG + (dF_*/d\varepsilon)$, and for the quadratic representation of the free energy [which corresponds to an isotropic body and takes into account condition (13)], $F_* = -2\mu\varepsilon_2$, it gives the modified Hooke's law

$$P = -qG + 2\mu\varepsilon. \tag{15}$$

Thus, the boundary-value problem with consideration for incompressibility and geometric nonlinearity contains Eqs. (1), (9), (13), and (15) and condition (10):

$$\begin{gathered} P\cdot\nabla + \mathbf{f} = 0, \quad P = -qG + 2\mu\varepsilon, \quad \mu = \text{const}, \quad \varepsilon_1 = \operatorname{tr}\varepsilon = 0, \\ 2\varepsilon = 2e + \omega\cdot\omega, \quad 2e = \nabla\mathbf{u} + \mathbf{u}\nabla, \quad 2\omega = \nabla\mathbf{u} - \mathbf{u}\nabla, \quad \mathbf{u}\big|_{\Sigma_u} = \mathbf{h}, \quad P\cdot\mathbf{n}\big|_{\Sigma_p} = \mathbf{p}. \end{gathered} \tag{16}$$

In contrast to the similar problem (11) for a compressible material, an additional unknown quantity, hydrostatic pressure, appears in relations (16); however, they also include an additional equation, i.e., the incompressibility condition. Therefore, the system of equations remains closed.

For plane deformation parallel to the bases of a cylindrical (or prismatic) body, the plane problem of [7] in which Eqs. (16) hold in a plane region S, which is the section of the body by the plane of deformation, and the boundary conditions are satisfied on the boundary L of this region, is basic. In the Cartesian x and y coordinates of the actual state which are defined on the plane of deformation, the relations of the plane problem follow from (16), and in the absence of bulk forces, they have the form

$$\begin{gathered} \frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} = 0, \quad \frac{\partial P_{xy}}{\partial x} + \frac{\partial P_{yy}}{\partial y} = 0, \quad P_{xx} = -q + 2\mu\varepsilon_{xx}, \quad P_{yy} = -q + 2\mu\varepsilon_{yy}, \\ P_{xy} = 2\mu\varepsilon_{xy}, \quad 2\varepsilon_{xx} = 2e_{xx} - \omega_{xy}^2, \quad 2\varepsilon_{yy} = 2e_{yy} - \omega_{xy}^2, \quad \varepsilon_{xy} = e_{xy}, \\ \varepsilon_{xx} + \varepsilon_{yy} = 0, \quad e_{xx} = \frac{\partial u_x}{\partial x}, \quad e_{yy} = \frac{\partial u_y}{\partial y}, \quad 2e_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}, \quad 2\omega_{xy} = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}; \end{gathered} \tag{17}$$

$$u_x\Big|_{L_u} = h_x(s),\ u_y\Big|_{L_u} = h_y(s),\ P_{xx}\frac{dy}{ds} - P_{xy}\frac{dx}{ds}\Big|_{L_p} = p_x(s),\ P_{xy}\frac{dy}{ds} - P_{yy}\frac{dx}{ds}\Big|_{L_p} = p_y(s). \tag{18}$$

Here the Cartesian components of the vectors and tensors are denoted by the same symbols as the quantities themselves, but with letter subscripts; L_u and L_p are parts of the boundary L on which displacements and stresses are defined, respectively; s is an arc of L; we take into account that the components of the normal to the contour L are representable in terms of the equations $x = x(s)$ and $y = y(s)$ by the formulas $n_x = dy/ds$ and $n_y = -dx/ds$.

Relations (17) permit obtaining first-order equations for stresses and rotation. Indeed, the incompressibility equation and the law of mechanical behavior give a representation of pressure in terms of stresses, $q = -(1/2)(P_{xx} + P_{yy})$, with allowance for which the strains are also expressed only in terms of stresses by the formulas $2\mu\varepsilon_{xx} = (1/2)(P_{xx} - P_{yy})$, $2\mu\varepsilon_{yy} = (1/2)(P_{yy} - P_{xx})$, and $2\mu\varepsilon_{xy} = P_{xy}$. Eliminating the displacements from the expressions for displacement gradients in terms of stresses and rotation,

$$2\mu\,\frac{\partial u_x}{\partial x} = \frac{1}{2}(P_{xx} - P_{yy}) + \mu\omega_{xy}^2, \quad 2\mu\,\frac{\partial u_x}{\partial y} = P_{xy} - 2\mu\omega_{xy},$$

$$2\mu\,\frac{\partial u_y}{\partial y} = \frac{1}{2}(P_{yy} - P_{xx}) + \mu\omega_{xy}^2, \quad 2\mu\,\frac{\partial u_y}{\partial x} = P_{xy} + 2\mu\omega_{xy},$$

we obtain compatibility equations for stresses and rotation. Supplementing them by the equilibrium equations [the first two equalities of (17)] gives the desired system

$$\Phi_1 = \frac{\partial(P_{xx} - P_{yy})}{\partial y} - 2\,\frac{\partial P_{xy}}{\partial x} + 4\mu\Big(\frac{\partial\omega_{xy}}{\partial x} + \omega_{xy}\,\frac{\partial\omega_{xy}}{\partial y}\Big) = 0,$$

$$\Phi_2 = \frac{\partial(P_{yy} - P_{xx})}{\partial x} - 2\,\frac{\partial P_{xy}}{\partial y} + 4\mu\Big(\omega_{xy}\,\frac{\partial\omega_{xy}}{\partial x} - \frac{\partial\omega_{xy}}{\partial y}\Big) = 0, \tag{19}$$

$$\Phi_3 = \frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} = 0, \quad \Phi_4 = \frac{\partial P_{xy}}{\partial x} + \frac{\partial P_{yy}}{\partial y} = 0.$$

Assuming that the force conditions (18) are defined over the entire boundary

$$P_{xx}\frac{dy}{ds} - P_{xy}\frac{dx}{ds}\Big|_L = p_x(s), \quad P_{xy}\frac{dy}{ds} - P_{yy}\frac{dx}{ds}\Big|_L = p_y(s), \tag{20}$$

we have the plane boundary-value problem (19) and (20) for stresses and rotation.

To investigate the type of system (19), for convenience, we introduce the following notation using subscripts for the desired functions and their arguments: $w_1 = P_{xx}$, $w_2 = P_{yy}$, $w_3 = P_{xy}$, and $w_4 = \omega_{xy}$; $x_1 = x$ and $x_2 = y$.

Let us consider, following [8], the characteristic determinant

$$\Delta_* = \det(A_{kl}), \quad A_{kl} = \sum_{m=1}^{2}\Big\{\partial\Phi_k/\partial\Big(\frac{\partial w_l}{\partial x_m}\Big)\Big\}\sigma_m \quad \Big(\sum_m \sigma_m^2 = 1, \quad k, l = \overline{1,4}\Big). \tag{21}$$

By virtue of (19) and (21), the determinant and its elements have the following values: $\Delta_* = 4\mu(\sigma_1^2 + \sigma_2^2)^2$ and $A_{11} = \sigma_2$, $A_{12} = -\sigma_2$, $A_{13} = -2\sigma_1$, $A_{14} = 4\mu(\sigma_1 + w_4\sigma_2)$, $A_{21} = -\sigma_1$, $A_{22} = \sigma_1$, $A_{23} = -2\sigma_2$. $A_{24} = 4\mu(w_4\sigma_1 - \sigma_2)$, $A_{31} = \sigma_1$, $A_{32} = 0$, $A_{33} = \sigma_2$, $A_{34} = 0$, $A_{41} = 0$, $A_{42} = \sigma_2$, $A_{43} = \sigma_1$, and $A_{44} = 0$.

Since $\mu > 0$, we have $\Delta_* > 0$. Therefore, the characteristic equation $\Delta_* = 0$ has no real roots. Thus, the quasi-linear system (19), as the corresponding system for linear elasticity, is elliptic, and the boundary problem (19) and (20) is correct for it.

Let us go over from the Cartesian to the complex coordinates z and $\bar z$ on the plane of deformation using the formulas

$$z = x + iy, \quad \bar z = x - iy, \quad 2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \quad 2\frac{\partial}{\partial \bar z} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

and consider the complex vector and tensor components (which are denoted in the same way as the quantities themselves but with numerical superscripts). They are related to the Cartesian components by the relations [5]

$$u^1 = \overline{u^2} = u_x + iu_y, \quad p^1 = \overline{p^2} = p_x + ip_y, \quad P^{11} = \overline{P^{22}} = P_{xx} - P_{yy} + 2iP_{xy},$$

$$P^{21} = P^{12} = P_{xx} + P_{yy}, \quad \varepsilon^{11} = \overline{\varepsilon^{22}} = \varepsilon_{xx} - \varepsilon_{yy} + 2i\varepsilon_{xy}, \quad \varepsilon^{21} = \varepsilon^{12} = \varepsilon_{xx} + \varepsilon_{yy}, \tag{22}$$

$$e^{11} = \overline{e^{22}} = e_{xx} - e_{yy} + 2ie_{xy}, \quad e^{21} = e^{12} = e_{xx} + e_{yy}, \quad \omega^{11} = \overline{\omega^{22}} = 0, \quad \omega^{21} = \overline{\omega^{12}} = 2i\omega_{xy}.$$

Then, Eqs. (17) and conditions (18) are written in compact form as

$$\frac{\partial P^{11}}{\partial z} + \frac{\partial P^{21}}{\partial \bar{z}} = 0, \; P^{11} = \overline{P^{22}} = 2\mu\varepsilon^{11}, \; P^{21} = P^{12} = -2q, \; \varepsilon^{11} = \overline{\varepsilon^{22}} = e^{11},$$

$$\varepsilon^{21} = \varepsilon^{12} = e^{21} + \frac{1}{4}(\omega^{21})^2, \; \varepsilon^{21} = 0, \; e^{11} = \overline{e^{22}} = 2\frac{\partial u^1}{\partial \bar{z}}, \tag{23}$$

$$e^{21} = \frac{\partial u^1}{\partial z} + \frac{\partial \overline{u^1}}{\partial \bar{z}}, \; \omega^{21} = \frac{\partial u^1}{\partial z} - \frac{\partial \overline{u^1}}{\partial \bar{z}};$$

$$u^1\Big|_{L_u} = h^1(s), \quad P^{21}\frac{dz}{ds} - P^{11}\frac{d\bar{z}}{ds}\Big|_{L_p} = 2ip^1(s). \tag{24}$$

Here one of the two conjugate complex equalities is written.

It follows from (23) that hydrostatic pressure and strain depend only on stresses:

$$q = -\frac{1}{2}P^{21}, \quad 2\mu\varepsilon^{11} = 2\mu\overline{\varepsilon^{22}} = P^{11}, \quad \varepsilon^{21} = 0. \tag{25}$$

The displacement gradients depend on stresses and rotation:

$$2\mu\frac{\partial u^1}{\partial z} = \mu\omega^{21}\Big(1 - \frac{1}{4}\omega^{21}\Big), \quad 2\mu\frac{\partial u^1}{\partial \bar{z}} = \frac{1}{2}P^{11}. \tag{26}$$

The compatibility condition for system (26) gives a compatibility consistency equation for stresses and rotation. Together with the equilibrium equation [the first equality of (23)] and the force condition (24) over the entire contour, it forms the following boundary-value problem for the stresses and rotation, which is a complex form of problem (19) and (20):

$$\frac{\partial P^{11}}{\partial z} = 2\mu\frac{\partial}{\partial \bar{z}}\Big[\omega^{21}\Big(1 - \frac{1}{4}\omega^{21}\Big)\Big], \quad \frac{\partial P^{11}}{\partial z} + \frac{\partial P^{21}}{\partial \bar{z}} = 0; \tag{27}$$

$$P^{21}\frac{dz}{ds} - P^{11}\frac{d\bar{z}}{ds}\Big|_L = 2ip^1(s). \tag{28}$$

System (27) admits full integration and representation of stresses and rotation in terms of potentials, and condition (28) leads to a boundary-value problem for the potentials. Indeed, eliminating P^{11} from (27) gives the equality

$$\frac{\partial}{\partial \bar{z}}\Big[P^{21} + 2\mu\omega^{21}\Big(1 - \frac{1}{4}\omega^{21}\Big)\Big] = 0,$$

from which, after integration, we obtain the relation $P^{21} + 2\mu\omega^{21}(1 - (1/4)\omega^{21}) = 4\varphi'(z)$, where $\varphi'(z)$ is an arbitrary complex potential. Separating the real and imaginary parts, we obtain the formulas $P^{21} - (\mu/2)(\omega^{21})^2 = 2(\varphi'(z) + \overline{\varphi'(z)})$ and $2\mu\omega^{21} = 2(\varphi'(z) - \overline{\varphi'(z)})$, which define the real stress P^{21} and the purely imaginary rotation ω^{21} as functions of the complex potential:

$$P^{21} = 2(\varphi'(z) + \overline{\varphi'(z)}) + \frac{1}{2\mu}(\varphi'(z) - \overline{\varphi'(z)})^2, \quad \mu\omega^{21} = \varphi'(z) - \overline{\varphi'(z)}. \tag{29}$$

In view of (29), the second equality of (27) becomes the equation for the complex stress P^{11},

$$\frac{\partial}{\partial z}\Big\{P^{11} + 2z\overline{\varphi''(z)} + \frac{1}{\mu}\overline{\varphi''(z)}(z\overline{\varphi'(z)} - \varphi(z))\Big\} = 0,$$

and, as a result of integration, it leads to the expression

$$P^{11} = -2(z\overline{\varphi''(z)} + \overline{\psi'(z)}) - \frac{1}{\mu}\overline{\varphi''(z)}(z\overline{\varphi'(z)} - \varphi(z)). \tag{30}$$

Here $\psi'(z)$ is an arbitrary function and the second complex potential. Thus, in the general solution (29) and (30) of Eqs. (27), stresses and rotation are expressed in terms of the complex potentials by nonlinear formulas. Substitution of the solution into condition (28) leads to the expression

$$\frac{d}{ds}\left\{\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \frac{1}{4\mu}\left(z\bar{\varphi}'^2(\bar{z}) - 2\varphi(z)\overline{\varphi'(z)} + \int \varphi'^2(z)\,dz\right)\right\}\bigg|_L = ip^1(s),$$

which, after integration along the contour, takes the form of a boundary problem for the potentials:

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \frac{1}{4\mu}N(\varphi(z), \overline{\varphi(z)})\Big|_L = g^1(s),$$

$$N(\varphi(z), \overline{\varphi(z)}) = z\bar{\varphi}'^2(\bar{z}) - 2\varphi(z)\overline{\varphi'(z)} + \int \varphi'^2(z)\,dz, \tag{31}$$

$$g^1(s) = i\int_0^s p^1(s)\,ds + C, \quad C = \text{const}.$$

Thus, in the geometrically nonlinear model of elasticity, the plane problem for stresses and rotation for an incompressible material reduces to the nonlinear boundary-value problem for complex potentials. The resulting potentials correspond to the complex stresses and rotation, and the latter correspond to the actual stresses and rotation defined by the inverted formulas (22).

It should be noted that the reduction of problem (27) and (28) to the potential problem (31) can also be performed by another method. Introduction of the real stress function $U(z, \bar{z})$ by the formulas

$$P^{11} = \overline{P^{22}} = -4\frac{\partial^2 U}{\partial \bar{z}^2}, \quad P^{21} = P^{12} = 4\frac{\partial^2 U}{\partial z\,\partial \bar{z}} \tag{32}$$

reduces problem (27) and (28) to the boundary-value problem for U and ω^{21}:

$$\frac{\partial}{\partial \bar{z}}\left[4\frac{\partial^2 U}{\partial z\,\partial \bar{z}} + 2\mu\omega^{21}\left(1 - \frac{1}{4}\omega^{21}\right)\right] = 0, \quad 2\frac{\partial U}{\partial \bar{z}}\bigg|_L = g^1(s). \tag{33}$$

The latter, in turn, reduces to the problem for the potentials. Indeed, integration of Eq. (33) leads to the relation

$$4\frac{\partial^2 U}{\partial z\,\partial \bar{z}} + 2\mu\omega^{21}(1 - \frac{1}{4}\omega^{21}) = 4\varphi'(z),$$

where $\varphi'(z)$ is an arbitrary function, and separation of the real and imaginary parts gives

$$4\frac{\partial^2 U}{\partial z\,\partial \bar{z}} - \frac{\mu}{2}(\omega^{21})^2 = 2(\varphi'(z) + \overline{\varphi'(z)}), \quad \mu\omega^{21} = \varphi'(z) - \overline{\varphi'(z)}. \tag{34}$$

The formula obtained for the rotation coincides with (29). With allowance for this expression, the first equality of (34) is transformed into the equation

$$2\frac{\partial^2 U}{\partial z\,\partial \bar{z}} = \varphi'(z) + \overline{\varphi'(z)} + \frac{1}{4\mu}(\varphi'^2(z) - 2\varphi'(z)\overline{\varphi'(z)} + \bar{\varphi}'^2(\bar{z})),$$

and integration of the latter with allowance for the reality of U gives the following nonlinear representation for the stress function in terms of the complex potentials $\varphi(z)$ and $\psi(z)$:

$$2U = \bar{z}\varphi(z) + z\overline{\varphi(z)} + \int \psi(z)\,dz + \overline{\int \psi(z)\,dz} + \frac{1}{4\mu}\left(\bar{z}\int \varphi'^2(z)\,dz + z\overline{\int \varphi'^2(z)\,dz} - 2\varphi(z)\overline{\varphi(z)}\right). \tag{35}$$

Now it is easy to see that expressions (29) and (30) for stresses in terms of potentials follow from (32) and (35), and the boundary-value problem for the potentials (31) follows from (35) and condition (33).

Using the representations of stresses and rotation in terms of potentials, we can consider the relations (26) as equations for displacements. The compatibility condition for the equations is satisfied: this is the first of Eqs. (27). Hence, the displacement differential has the form

$$d(2\mu u^1) = \mu\omega^{21}\Big(1 - \frac{1}{4}\,\omega^{21}\Big)\,dz + \frac{1}{2}\,P^{11}d\bar z = d\Big[\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - \frac{1}{4\mu}\,N(\varphi(z), \overline{\varphi(z)})\Big],$$

where N is defined by formula (31). Hence, the displacement itself is represented in terms of potentials by the nonlinear expression

$$2\mu u^1 = \varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - \frac{1}{4\mu}\,N(\varphi(z), \overline{\varphi(z)}) + \text{const.} \tag{36}$$

The additive constant entering into this expression corresponds to the rigid translation of the body; it is insignificant and can be omitted.

Finally, using relations (25) and (29), we can express the hydrostatic pressure in terms of potentials:

$$q = -(\varphi'(z) + \overline{\varphi'(z)}) - \frac{1}{4\mu}\,(\varphi'(z) - \overline{\varphi'(z)})^2. \tag{37}$$

Comparison of expressions (29), (30), (36), and (31) for stresses, rotation, displacement, and the boundary condition for potentials for incompressible materials with the corresponding expressions obtained in [5] for compressible materials using the same nonlinear model and the same boundary conditions,

$$P^{11} = -2(z\overline{\varphi''(z)} + \overline{\psi'(z)}) - 2\,\frac{1-\nu}{\mu}\,\overline{\varphi''(z)}(z\overline{\varphi'(z)} - \varphi(z)),$$

$$P^{21} = 2(\varphi'(z) + \overline{\varphi'(z)}) + \frac{1-\nu}{\mu}\,(\varphi'(z) - \overline{\varphi'(z)})^2, \quad \mu\omega^{21} = 2(1-\nu)(\varphi'(z) - \overline{\varphi'(z)}),$$

$$2\mu u^1 = (3 - 4\nu)\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - \frac{1-\nu}{2\mu}\,N(\varphi(z), \overline{\varphi(z)}),$$

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \frac{1-\nu}{2\mu}\,N(\varphi(z), \overline{\varphi(z)})\Big|_L = g^1(s),$$

where ν is the Poisson coefficient, shows that the former relations follow from the latter for $\nu = 1/2$. This allows one to consider an incompressible nonlinear material as a limiting compressible nonlinear material for which the Poisson coefficient is equal to 1/2.

As was established in [5], for weak (in comparison with μ) loads, the formulas of linear elasticity follow from the formulas of nonlinear theory in which nonlinear terms are omitted. Applying this method to (29)-(31), (36), and (37), we obtain the following relations for a linear incompressible material:

$$P^{11} = -2(z\overline{\varphi''(z)} + \overline{\psi'(z)}), \quad P^{21} = 2(\varphi'(z) + \overline{\varphi'(z)}), \quad \mu\omega^{21} = \varphi'(z) - \overline{\varphi'(z)},$$

$$2\mu u^1 = \varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \quad q = -\varphi'(z) - \overline{\varphi'(z)}, \quad \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}\Big|_L = g^1(s).$$

If the displacement vector or the stress vector are specified over the entire boundary of a body, the boundary problem for potentials has the form of one of the complex conditions

$$\begin{gathered} \varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - (1/4\mu)\,N(\varphi(z), \overline{\varphi(z)})\Big|_L = 2\mu h^1(s), \\ \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + (1/4\mu)\,N(\varphi(z), \overline{\varphi(z)})\Big|_L = g^1(s). \end{gathered} \tag{38}$$

These problems are nonlinear and of the same type. In some of cases, an approximate solution can be found by expansion in terms of a small parameter.

Let us consider the problem of stresses under weak-load conditions when the characteristic pressure P_0 is small in comparison with the shear modulus μ. Then, the dimensionless stress $\sigma = P_0/\mu$ is a small parameter ($\sigma \ll 1$). Expanding the complex potentials in terms of this parameter

$$\varphi(z) = \sum_0^\infty \sigma^\nu \varphi_\nu(z), \quad \psi(z) = \sum_0^\infty \sigma^\nu \psi_\nu(z) \tag{39}$$

and substituting them into the second condition (38), we find that the latter reduces to a sequence of boundary-value problems for the component potentials φ_ν and ψ_ν:

$$\varphi_\nu(z) + z\overline{\varphi'_\nu(z)} + \overline{\psi_\nu(z)}\Big|_L = v_\nu \quad (\nu = 0, 1, 2, \ldots). \tag{40}$$

Here $v_0 = g^1$, $v_{\nu+1} = -(1/4P_0)\,N_\nu$, and

$$N_0 = z\bar{\varphi}_0'^2 - 2\varphi_0\bar{\varphi}_0' + \int \varphi_0'^2\,dz;$$

$$N_1 = 2(z\bar{\varphi}_0'\bar{\varphi}_1' - \varphi_0\bar{\varphi}_1' - \varphi_1\bar{\varphi}_0' + \int \varphi_0'\varphi_1'\,dz);$$

$$N_2 = z(\bar{\varphi}_1'^2 + 2\bar{\varphi}_0'\bar{\varphi}_2') - 2(\varphi_0\bar{\varphi}_2' + \varphi_1\bar{\varphi}_1' + \varphi_2\bar{\varphi}_0') + \int(\varphi_1'^2 + 2\varphi_0'\varphi_2')\,dz;$$

$$\ldots,$$

and the zero approximation corresponds to the linear elastic problem. In the problem for the νth approximation, the right side of the boundary condition is defined by previous approximations and is thereby known. Therefore, each pair of potentials φ_ν and $\bar{\psi}_\nu$ is found, according to (40), from the boundary-value problem of linear elasticity; the methods for solving the latter are known [7].

Substitution of expansions (39) into formulas (29), (30), (36), and (37) gives the following expansions of stresses, rotation, displacement, and pressure in terms of parameters:

$$P^{11} = \sum_0^\infty \sigma^\nu P_\nu^{11},\ P^{21} = \sum_0^\infty \sigma^\nu P_\nu^{21},\ \mu\omega^{21} = \mu\sum_0^\infty \sigma^\nu \omega_\nu^{21},\ 2\mu u^1 = 2\mu\sum_0^\infty \sigma^\nu u_\nu^1,\ q = \sum_0^\infty \sigma^\nu q_\nu.$$

Here the zero approximation is defined by the potentials of linear elasticity, and the νth approximation, by the νth and all previous component potentials.

Linear problems for the potentials can be obtained by setting boundary values different from those in (38). Let us consider two cases.

Let the displacement $h_x(s)$ and the stress $p_x(s)$ [and, hence, $g_x = \int p_x(s)\,ds$] be specified on the entire boundary, and let the displacement h_y^0 be specified at the boundary point O. Then, the potentials should be found from the two real conditions

$$\mathrm{Re}\left\{\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \frac{1}{4\mu}N(\varphi,\bar{\varphi})\right\}\Big|_L = g_x(s),$$

$$\mathrm{Re}\left\{\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - \frac{1}{4\mu}N(\varphi,\bar{\varphi})\right\}\Big|_L = 2\mu h_x(s),$$

which can be transformed into the Dirichlet problems for each potential:

$$\mathrm{Re}\,\varphi(z)\Big|_L = k(s), \quad \mathrm{Re}\,\psi(z)\Big|_L = l(s). \tag{41}$$

Here $k = (1/2)g_x + \mu h_x$ and $l = g_x - \mathrm{Re}\,\{\varphi(z) + z\overline{\varphi'(z)} + (1/4\mu)N(\varphi,\bar{\varphi})\}\big|_L$.

Successive solution of these problems (wherein the solution to the first problem determines the right side of the expression for the second) gives the potential values $\varphi(z)$ and $\psi(z)$ with accuracy to the additive constants iA_y and iB_y, respectively [9]. The constant B_y does not affect the stresses and rotation and in the expression for displacement, it is additive; it is insignificant and can be taken equal to zero. The constant A_y is significant; it is specified by the condition $\mathrm{Im}\,u^1(z_0, \bar{z}_0) = h_y^0$ at point O.

If the rotation $\omega_{xy}(s)$ is specified on the entire boundary, and the stress $p_x(s)$ and the quantities h_x^0 and p_y^0 are specified at the boundary point O, then, according to (29), the potential $i\varphi'(z)$ is determined from the Dirichlet problem

$$\mathrm{Re}\,(i\varphi'(z))\Big|_L = -\mu\omega_{xy}(s) \tag{42}$$

with accuracy to the constant iA_y, and the potential $\varphi(z)$ (calculated by quadrature) is determined with accuracy to the constants A_y and $C = C_x + iC_y$. The potential $\psi(z)$ is found from the second condition of

(41) with accuracy to an insignificant constant. The constants entering into the potentials are determined from the conditions

$$\operatorname{Re}\varphi(z_0) = \frac{1}{2} g_x^0 + \mu h_x^0, \quad P^{21}(z_0, \bar{z}_0)\left(\frac{dz}{ds}\right)_0 - P^{11}(z_0, \bar{z}_0)\left(\frac{d\bar{z}}{ds}\right)_0 = 2i(p_x^0 + ip_y^0).$$

It should be noted that in the case of a simply connected region, it can be conformally mapped onto the interior of a circle of unit radius. Then, the Dirichlet problems (41) and (42) will be formulated on its circumference, and their solutions can be represented by the Schwartz formula [9].

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